

A Bounded Derivation Method for the Maximum Likelihood Estimation on the Parameters of Weibull Distribution

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Abstract—For the basic maximum likelihood estimating function of the two parameters Weibull distribution, a simple proof on its global monotonicity is given to ensure the existence and uniqueness of its solution. The boundary of the function's first-order derivative is defined based on its scale-free property. With a bounded derivative, the possible range of the root of this function can be determined. A novel root-finding algorithm employing these established results is proposed accordingly, its convergence is proved analytically as well. Compared with other typical algorithms for this problem, the efficiency of the proposed algorithm is also demonstrated by numerical experiments.

Index Terms—Two parameter Weibull Distribution; Maximum Likelihood Estimation; Global Monotonicity; Scale-free Property.

I. INTRODUCTION

The Weibull distribution [1] is an important distribution in reliability and maintainability analysis. The estimation on its parameters has been widely discussed, and there are several methodological categories for this parameter estimation issue [2] [3] [4]. For example, the graphic methods [5], transcendental equation-solving method applying bifurcation algorithm [4], maximum likelihood estimation (MLE) method [6] [7] [8]. The graphic methods, such as Weibull probability plotting (WPP) [9], are straight forward, but can not give a precise estimation. The transcendental equation-solving method has a closed form, it can avoid the computing expense in the iterative computation of the raw data, but in order to solve the transcendental equation, advanced mathematical techniques are required.

In the MLE-based methods, as the basic estimating equation is not in closed form, it can be solved only numerically. There are several typical MLE-based methods for solving this equation, such as the secant method, the bisection method and the Newton-Raphson method. However, in both the secant method and the bisection method, the convergence rates are very low; in the Newton-Raphson method, it has to compute both the basic estimating function and its derivative [10] at each iterative step. In some cases, the Newton-Raphson method cannot ensure convergence [11]. Furthermore, these above MLE-based methods require either initial values or trial computation of the estimated parameters.

In [12], the author claims that the existence and uniqueness of the solution of the basic estimating equation under MLE method cannot be assured. However, in [13] [14], based on Cauchy-Schwarz inequality, the authors have given similar proofs of the existence and uniqueness on the solution of the MLE-based estimator. In this paper, we present a straight forward proof by mathematics induction, which is much simpler than both the proofs given in [13] [14].

For the basic estimating function, we have proved the scale free property of its first order derivative, the boundary of this derivative can thus be defined. Moreover, with a bounded derivative, at each iterative step, the possible range of the root of the basic estimating function can be determined. We thus propose a novel MLE-based root-finding algorithm based on these properties, its computational efficiency and advantages are well demonstrated by numerical experiments.

The remainder of this paper is organized as follows. In Section II, a proof on the monotonicity of the basic estimating function is given. In Section III, the scale-free property of the first order derivative of this function is proved, the boundaries of the function itself and its first order derivative are defined. The feasible range of its root thus can be determined. In Section IV, by employing these proved results, a novel root-finding algorithm is hence designed. The convergence of the proposed algorithm is proved. In Section V, the performance of this proposed root-finding algorithm are demonstrated by numerical experiments. Conclusions are given in VI.

II. THE GLOBAL MONOTONICITY OF THE BASIC ESTIMATING FUNCTION

A. The Basic Estimating Function for the Two Parameter Weibull Distribution

The density function of the two parameter Weibull distribution is:

$$f(x) = \left(\frac{k}{\lambda}\right) \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \quad (x \geq 0, k > 0, \lambda > 0) \quad (1)$$

For a sampled data of n observations with the above Eqn.1 as the applicable density function, its likelihood function is

$$L(x_1, \dots, x_n; k, \lambda) = \prod_{i=1}^n \left(\frac{k}{\lambda}\right) \left(\frac{x_i}{\lambda}\right)^{k-1} e^{-\left(\frac{x_i}{\lambda}\right)^k} \quad (2)$$

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By MLE method, the following equations can be obtained:

$$\frac{\partial \ln L}{\partial k} = \frac{n}{k} + \sum_{i=1}^n \ln x_i - \frac{1}{\lambda^k} \sum_{i=1}^n x_i^k \ln x_i = 0, \quad (3)$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{k}{\lambda} \left(-n + \frac{1}{\lambda^k} \sum_{i=1}^n x_i^k \right) = 0. \quad (4)$$

by eliminating λ , we get

$$\frac{\sum_{i=1}^n x_i^k \ln x_i}{\sum_{i=1}^n x_i^k} - \frac{1}{k} = \frac{1}{n} \sum_{i=1}^n \ln x_i \quad (5)$$

by the above Eqn.5, we can get the value of k by related numerical algorithms. With k determined, λ can be calculated by Eqn.4 as

$$\hat{\lambda} = \left(\frac{\sum_{i=1}^n x_i^{\hat{k}}}{n} \right)^{\frac{1}{\hat{k}}} = \sqrt[\hat{k}]{\frac{1}{n} \sum_{i=1}^n x_i^{\hat{k}}} \quad (6)$$

we can therefore calculate both k and λ . Here \hat{k} ($\hat{\lambda}$) refers to maximum likelihood estimators for parameter k (λ).

By Eqn.5, we can define the basic estimating function $F(k)$ as

$$F(k) = \frac{\sum_{i=1}^n x_i^k \ln x_i}{\sum_{i=1}^n x_i^k} - \frac{1}{n} \sum_{i=1}^n \ln x_i - \frac{1}{k} \quad (7)$$

Thus $F(k) = 0$ here is defined as the basic estimating equation. By mathematics induction, with $n \geq 1$ and $k > 0$, a simple proof on the global monotonicity of $F(k)$, i.e., $\frac{\partial F(k)}{\partial k} > 0$ can be given in the following section.

B. Proof on the global monotonicity of $F(k)$

Since

$$\frac{\partial F(k)}{\partial k} = \frac{1}{k^2} + \left\{ \sum_{i=1}^n x_i^k \ln^2 x_i \sum_{i=1}^n x_i^k - \left(\sum_{i=1}^n x_i^k \ln x_i \right)^2 \right\} \left(\sum_{i=1}^n x_i^k \right)^{-2} \quad (8)$$

to prove $\frac{\partial F(k)}{\partial k} > 0$, we need only to prove that for $k > 0$ and any $x_i \in R^+$,

$$P_1(x_i, n, k) = \sum_{i=1}^n x_i^k \ln^2 x_i \sum_{i=1}^n x_i^k - \left(\sum_{i=1}^n x_i^k \ln x_i \right)^2 \geq 0$$

for $n = 1$

$$P_1(x_i, 1, k) = x_1^k \ln^2 x_1 \cdot x_1^k - x_1^{2k} \cdot \ln^2 x_1 = 0 \quad (9)$$

for $n = 2$

$$P_1(x_i, 2, k) = x_1^k x_2^k (\ln x_2 - \ln x_1)^2 \geq 0 \quad (10)$$

for $n = m - 1$ ($m \geq 3$, $m \in N^+$), suppose

$$P_1(x_i, m - 1, k) \geq 0 \quad (11)$$

then for $n = m$

$$P_1(x_i, m, k) = P_1(x_i, m - 1, k) + x_m^k \sum_{i=1}^{m-1} x_i^k (\ln x_m - \ln x_i)^2 \geq 0 \quad (12)$$

Therefore $\frac{\partial F(k)}{\partial k} > 0$, and function $F(k)$ is global monotonic. With the global monotonicity of $F(k)$, the existence and uniqueness of the root of $F(k) = 0$ can be assured.

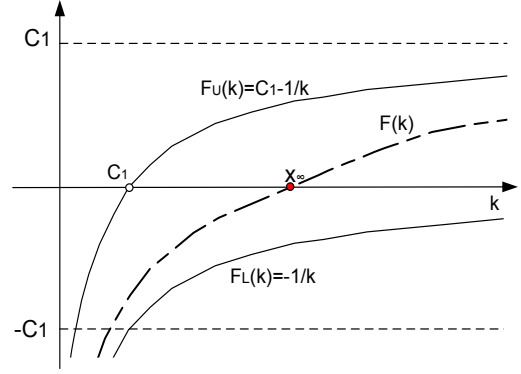


Fig. 1. The boundaries of $F(k)$: $-\frac{1}{k} < F(k) < -\frac{1}{k} + C_1$. Here $C_1 = \frac{1}{n} \sum_{i=1}^{n-1} \ln \frac{x_{max}}{x_i}$, $k > 0$.

III. BOUNDARIES OF THE BASIC ESTIMATING FUNCTION AND ITS FIRST-ORDER DERIVATIVE

A. Boundaries of the Basic Estimating Function $F(k)$

Let

$$G(k) = \frac{\sum_{i=1}^n x_i^k \ln x_i}{\sum_{i=1}^n x_i^k} - \frac{1}{n} \sum_{i=1}^n \ln x_i \quad (13)$$

as

$$\begin{cases} G(0) = 0 \\ G(+\infty) = \frac{1}{n} \sum_{i=1}^{n-1} \ln \frac{x_{max}}{x_i} > 0 \end{cases}$$

also because

$$\frac{\partial G(k)}{\partial k} = P_1(x_i, n, k) \left(\sum_{i=1}^n x_i^k \right)^{-2} \geq 0, \quad (14)$$

thus $\forall k > 0$

$$-\frac{1}{k} \leq F(k) \leq C_1 - \frac{1}{k} \quad (15)$$

here

$$C_1 = G(+\infty) = \frac{1}{n} \sum_{i=1}^{n-1} \ln \frac{x_{max}}{x_i} \quad (16)$$

we define the lower boundary of $F(k)$ is curve: $F_2(k) = -\frac{1}{k}$, the upper boundary of $F(k)$ is curve: $F_1(k) = -\frac{1}{k} + C_1$. Then the boundary curves of $F(k)$ can be seen in Fig.1.

B. Boundaries of the First Order derivative: $\frac{\partial F(k)}{\partial k}$

Let

$$H(n, x_i, k) = \left\{ \sum_{i=1}^n x_i^k \ln^2 x_i \sum_{i=1}^n x_i^k - \sum_{i=1}^n x_i^k \ln x_i \sum_{i=1}^n x_i^k \ln x_i \right\} \left(\sum_{i=1}^n x_i^k \right)^{-2} \quad (17)$$

Theorem A: $\forall n \in N^+, \forall x_i > 0$, and $\forall k > 0$, $\frac{\partial F(k)}{\partial k} \in [k^{-2}, k^{-2} + \ln^2(\frac{x_{max}}{x_{min}})]$.

PROOF: For a certain $\lambda_j > 0$ satisfying $\ln(\lambda_j \cdot x_i) \geq 0$,

$$\begin{aligned} H(n, x_i, k) &= H(n, \lambda_j \cdot x_i, k) \\ &\leq \max\{H(n, \lambda_j \cdot x_i, k)\} \\ &= \ln^2(\lambda_j \cdot x_{max}) - \ln^2(\lambda_j \cdot x_{min}) \\ &= \ln \frac{x_{max}}{x_{min}} \ln(\lambda_j^2 \cdot x_{max} \cdot x_{min}) \end{aligned} \quad (18)$$

Since to satisfy

$$\begin{cases} \ln(\lambda_j \cdot x_i) \geq 0 \\ \ln(\lambda_j^2 \cdot x_{max} \cdot x_{min}) \geq 0 \end{cases}$$

the minimum value of λ_j is x_{min}^{-1} , i.e., $\lambda_j \in [x_{min}^{-1}, +\infty]$.

let $\lambda = \gamma \cdot x_{min}^{-1}$ ($\gamma \geq 1$),

$$\max_{\substack{n \in N^+, x_i > 0, \\ k > 0, \gamma \geq 1}} \{H(n, \lambda \cdot x_i, k)\} = \ln\left(\frac{x_{max}}{x_{min}}\right) \ln\left(\gamma^2 \frac{x_{max}}{x_{min}}\right) \quad (19)$$

therefore the supermum value of $H(n, x_i, k)$ is

$$\begin{aligned} \sup\{H(n, x_i, k)\} &= \min\left\{\max_{\substack{x_i > 0, n \in N^+, \\ k > 0, \gamma \geq 1}} \{H(n, \lambda \cdot x_i, k)\}\right\} \\ &= \min_{\gamma \geq 1} \left\{ \ln\left(\frac{x_{max}}{x_{min}}\right) [\ln \gamma^2 + \ln\left(\frac{x_{max}}{x_{min}}\right)] \right\} \\ &= \ln^2\left(\frac{x_{max}}{x_{min}}\right) \end{aligned}$$

Hence

$$H(n, x_i, k) \leq \ln^2\left(\frac{x_{max}}{x_{min}}\right) \quad (20)$$

It is proved in Section II-B, the infimum value of $H(n, \lambda x_i, k)$ is:

$$\inf\{H(n, x_i, k)\} = 0 \quad (21)$$

with $\frac{\partial F(k)}{\partial k} = H(n, x_i, k) + \frac{1}{k^2}$, therefore

$$\frac{1}{k^2} \leq \frac{\partial F(k)}{\partial k} \leq C_2 + \frac{1}{k^2} \quad (22)$$

here $C_2 = \ln^2\left(\frac{x_{max}}{x_{min}}\right)$. \square

With the boundary of $\frac{\partial F(k)}{\partial k}$ defined, the feasible solution range of $F(k) = 0$ can be determined.

C. Possible Range of the Final Solution of Equation $F(k) = 0$

The basic estimating function $F(k)$ can be written as

$$F(k) = \int_{k_0}^k F'(\tau) d\tau$$

According to Theorem A,

$$\begin{cases} \int_{k_0}^k \tau^{-2} d\tau < \int_{k_0}^k F'(\tau) d\tau < \int_{k_0}^k [C_2 + \tau^{-2}] d\tau & (k > k_0) \\ \int_{k_0}^k \tau^{-2} d\tau > \int_{k_0}^k F'(\tau) d\tau > \int_{k_0}^k [C_2 + \tau^{-2}] d\tau & (k < k_0) \end{cases}$$

let denote

$$\begin{cases} F_L(k, k_0) = \int_{k_0}^k \tau^{-2} d\tau \\ F_U(k, k_0) = \int_{k_0}^k [C_2 + \tau^{-2}] d\tau \end{cases}$$

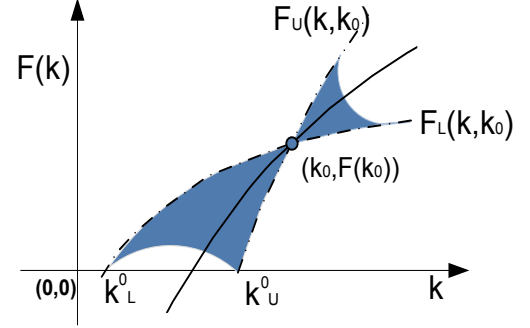


Fig. 2. The two boundary curves of $F(k)$: $F_L(k, k_0)$ and $F_U(k, k_0)$, which can determine the feasible range of the solution of $F(k) = 0$.

thus

$$\begin{cases} F_L(k, k_0) < F(k) < F_U(k, k_0) & (k > k_0) \\ F_L(k, k_0) > F(k) > F_U(k, k_0) & (k < k_0) \end{cases}$$

Here $F_U(k, k_0)$ and $F_L(k, k_0)$ can be seen the boundary curves of $F(k)$ at point $(k_0, F(k_0))$ (see Fig.2). With knowing $F(k_0)$, the two boundary curves of $F(k)$ can be determined as,

$$\begin{cases} F_L(k, k_0) = k_0^{-1} + F(k_0) - \frac{1}{k} \\ F_U(k, k_0) = C_2 k - \frac{1}{k} + (F(k_0) - C_2 k_0 + \frac{1}{k_0}) \end{cases}$$

We can see that both $F_U(k, k_0)$ and $F_L(k, k_0)$ are algebraic functions with simple mathematical forms.

Suppose $F_U(k, k_0)$ cuts k -axis at point k_U^0 , $F_L(k, k_0)$ cuts k -axis at point k_L^0 , in Fig.2, it is obvious that the root of equation $F(k, k_0) = 0$ must exist in $[k_L^0, k_U^0]$ (if $k_L^0 < k_U^0$) or in $[k_U^0, k_L^0]$ (if $k_L^0 > k_U^0$).

A detailed algorithm employing the properties in Section III-A, III-B and III-C will be proposed in the following section, and a proof on its convergence will be demonstrated as well.

IV. ROOT-FINDING ALGORITHM DESIGN

To numerically get the solution of $F(k) = 0$, an intuitive idea is to calculate $F(k_1)$ at $k_1 = \frac{1}{2}(k_L^0 + k_U^0)$, to further find out a narrower interval $[k_U^1, k_L^1]$ or $[k_L^1, k_U^1]$. By the same procedure, at $k_{i+1} = \frac{1}{2}(k_L^i + k_U^i)$ ($i \rightarrow +\infty$), the final solution of $F(k) = 0$ can be iteratively approximated.

From the mathematical form of the basic estimating function $F(k)$, we can see that for large n, k and x_i , the computational complexity for $F(k)$ is very high, thus the calculation time of $F(k)$ is an important index in measuring the efficiency of a root-finding algorithm for $F(k) = 0$. A novel root-finding algorithm has been designed based on these properties proved in Section II, III and III-C.

A. Convergence of the Bounded derivative Algorithm

As stated before, a straight forward method employing this idea is to calculate the final solution iteratively. The feasibility of this method can be ensured by the following theorems.

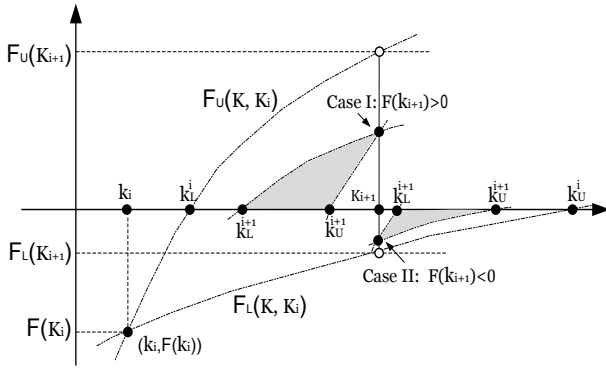


Fig. 3. Estimation on the convergence rate of the proposed algorithm with knowing $F(k_i)$ and $F(k_{i+1})$, here $F_L(k, k_i)$ and $F_U(k, k_i)$ refer to the boundary functions of $F(k)$ at point $(k_i, F(k_i))$, $k_{i+1} = \frac{1}{2}(k_L^i + k_U^i)$.

Lemma. For any point $(k_i, F(k_i))$ of function $F(k)$, the existence and uniqueness of the roots of $F_U(k, k_i) = 0$ and $F_L(k, k_i) = 0$ can be assured, therefore the interval $[k_L^i, k_U^i]$ ($i \geq 1$) that covers the root of $F(k) = 0$ always exists.

PROOF: (1) By Inequality 15, we know that $\forall k > 0$, $F(k) > -\frac{1}{k}$. The root of equation $F_L(k, k_0) = k_0^{-1} + F(k_0) - \frac{1}{k} = 0$ is

$$k_L^0 = \frac{1}{F(k_0) - (-k_0^{-1})} > 0 \quad (23)$$

which is reasonable.

(2) Since $F_U(k, k_0) = 0$ can be rewritten as

$$C_2 k^2 + (F(k_0) - C_2 k_0 + \frac{1}{k_0})k - 1 = 0 \quad (24)$$

Since here

$$\begin{cases} \Delta = B^2 - 4AC = (F(k_0) - C_2 k_0 + \frac{1}{k_0})^2 + 4C_2 > 0 \\ C = -1 < 0 \end{cases}$$

it is obvious that $F_U(k, k_0) = 0$ always has two roots with different signatures. To satisfy $k > 0$, only the positive root should be preserved.

Therefore, the existence of the interval $[k_L^i, k_U^i]$ ($i \geq 1$) can be assured. \square

Theorem B. Let $[k_L^i, k_U^i]$ denote the feasible interval of the solution of $F(k) = 0$ at point $(k_i, F(k_i))$, $[k_L^{i+1}, k_U^{i+1}]$ denote the feasible interval at point $(k_{i+1}, F(k_{i+1}))$, with $k_{i+1} = \frac{1}{2}(k_L^i + k_U^i)$, then the convergence rate $\gamma = \frac{\|k_L^{i+1} - k_U^{i+1}\|}{\|k_L^i - k_U^i\|} < \frac{1}{2}$. Here $k_L^{i+1} = \max(k_L^i, k_L^{i+1})$, $k_U^{i+1} = \min(k_U^i, k_U^{i+1})$, $i \in N^+$.

PROOF: With knowing point $(k_i, F(k_i))$ ($i \geq 1$), we can deduce the boundary curves: $F_L(k, k_i)$ and $F_U(k, k_i)$, and the related feasible interval $[k_L^{i+1}, k_U^{i+1}]$ can be determined accordingly (as shown in Fig.3).

Since if $F(k_{i+1}) = 0$, then K is the root we are looking for. Thus here there are only two situations for $F(k_{i+1})$: $F(k_{i+1}) > 0$ and $F(k_{i+1}) < 0$.

Case I: $F(k_{i+1}) > 0$

Fig. 4. The flow chart for the combined algorithm. Here δ_1 is a scale threshold, δ_2 is the approximating precision or the computing halt criterion.

With knowing $F(k_{i+1})$ ($i \geq 1$), the boundary curves $F_L(k, k_{i+1})$ and $F_U(k, k_{i+1})$ at point $(k_{i+1}, F(k_{i+1}))$ can be determined, and so is the related feasible interval $[k_L^{i+1}, k_U^{i+1}]$.

As in this case, by Fig.3, it is obvious that $k_U^{i+1} < K_{i+1}$, then the convergence rate

$$\gamma = \frac{k_U^{i+1} - k_L^{i+1}}{k_U^i - k_L^i} \leq \frac{k_U^{i+1} - k_L^i}{k_U^i - k_L^i} < \frac{k_{i+1} - k_L^i}{k_U^i - k_L^i} = \frac{1}{2} \quad (25)$$

Case II: $F(k_{i+1}) < 0$

Similarly, as shown in Fig.3, in this case, it is obvious that $k_L^{i+1} > K_{i+1}$, then the convergence rate

$$\gamma = \frac{k_U^{i+1} - k_L^{i+1}}{k_U^i - k_L^i} \leq \frac{k_U^i - k_L^{i+1}}{k_U^i - k_L^i} < \frac{k_U^i - k_{i+1}}{k_U^i - k_L^i} = \frac{1}{2} \quad (26)$$

Thus the proposed algorithm is convergence, and its convergence rate $\gamma < \frac{1}{2}$. \square

B. Improved Algorithm Combining the Secant Method and the Bounded derivative Method

It is obvious that at large scale the bounded derivative algorithm can converge rapidly, at least at the same rate as the bisection method. While in a small scale, especially in the linearizable neighborhood around the final solution point of the basic estimating equation, the secant method method has a better convergence rate. Therefore it is intuitive to combine both the two methods together. The flow chart for the combined algorithm can be seen in Fig.4.

As shown in the flow chart in Fig.4, in the combined algorithm, at a large scale, at each iterative step, we calculate $F(k_i)$ one time, then with only basic algebraic calculation, we find the possible range of the solution, and then narrow down it to a smaller range at next step. When the possible range of the root comes to a small scale, where the linearization part of the basic estimating function dominates, the combined algorithm will switch to the secant method, since it has a better performance in this case.

Moreover, compared with the secant method, the bisection method and the Newton-Raphson method, which requires initial values of $F(k)$, no initial value is required in the combined method.

V. NUMERICAL EXAMPLES

A. General Cases

With MATLAB, 1000 times of numerical experiments have been taken to study the performance of this proposed algorithm. In these simulations, the shape parameter of Weibull distribution $k \sim U(0, 40]$, the scale parameter $\lambda \sim U(0, 40]$, the number of data $N \sim U[2, 1000]$, here U refers to Uniform Distribution.

Compared with the secant method, the bisection method as well as the bi-secant method (which has combined the secant and the bisection method), the computing complexity (here refers to the calculation time of $F(k)$) of each method under various approximating precision ϵ can be seen in the following Table I.

Table I: Averaged calculation times of $F(k)$

Appro-precision ϵ	Secant method	Bisection method[6]	Bi-secant Method	Proposed method
10^{-1}	9.42	5.21	3.12	1.68
10^{-2}	30.22	6.33	4.29	2.58
10^{-3}	88.5	11.22	9.39	4.28
10^{-4}	130.6	14.19	11.51	5.05

B. A Constructed Case

The advantage of the proposed algorithm can also be verified by a concrete case, which is generated from a Weibull Distribution. The sampled data is in Table II, the comparison of the performance of different methods can be seen in Table III.

Table II: Sampled Data from a Weibull Distribution

2.6144	4.1834	4.3258	4.3496	4.3740	4.4006
3.2073	4.2573	4.3273	4.3544	4.3828	4.4051
3.9800	4.2884	4.3334	4.3646	4.3873	4.4123
4.1767	4.3150	4.3403	4.3698	4.3959	4.4194
4.4317	4.4919	4.4448	4.5082	4.4623	4.5439
4.4756	4.5715				

Table III: Calculation times of $F(k)$ of various methods

Appro-precision ϵ	Secant method	Bisection method	Bi-Secant Method	Proposed method
10^{-1}	4	4	3	1
10^{-2}	79	6	3	1
10^{-3}	178	10	3	2
10^{-4}	271	13	21	3
10^{-6}	459	20	22	4
10^{-10}	833	33	23	5
10^{-14}	1206	46	24	6

VI. CONCLUSION

In this paper, to assure the uniqueness and existence of the solution of the basic estimating function, we have proved its global monotonicity in a very simple way. With proving the

scale-free property of this function's first order derivative, the possible range of its solution at each iterative step can be determined. Based on these properties, a novel root-finding algorithm is proposed in this paper. Its efficiency has been demonstrated by numerical experiments.

For future work, this method can be extended to type I and type II data censoring cases.

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